Nonlinear density wave theory for the spiral structure of galaxies

S. Kondoh and R. Teramoto
Graduate School of Engineering, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

Z. Yoshida
Graduate School of Frontier Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

(Received 22 October 1999)

The theory of nonlinear waves for plasmas has been applied to the analysis of the density wave theory of galaxies which are many-body systems of gravity. A nonlinear Schrödinger equation has been derived by applying the reductive perturbation method on the fluid equations that describe the behavior of infinitesimally thin disk galaxies. Their spiral arms are characterized by a soliton and explained as a pattern of a propagating nonlinear density wave.

PACS number(s): 98.52.Nr, 52.35.Mw

I. INTRODUCTION

A galaxy is a many-body system composed of stars interacting with each other through a long range force, gravity [1]. There are a lot of common characteristics between the physics of galaxies and plasmas, which are systems of charged particles interacting through the Lorentz force. Macroscopic description of these systems can be casted in a fluid model including internal forces which bring about collective behavior such as waves or instabilities [1,2].

The spiral of a galaxy has been explained by the density wave theory [1,3,4] as a pattern of a rotating density wave on a disk. This theory resolves the winding, i.e., the spiral arms must be wound up far beyond the observed structure in normal ages of galaxies if we assume a simple rotational deformation ansatz [1]. Lin and Shu [3] treated galaxies as compressible fluids and showed that the linearized fluid equations for an infinitesimally thin disk galaxy have a spiral wave solution that is proportional to $\exp[\omega t - m\theta + \lambda f(r)]$, where ω is the frequency of the wave, m is the azimuthal wave number (which is also the number of arms of the galaxy), λ is a constant ($\lambda \gg 1$), $f(r)$ is the phase factor of the wave in the radial direction (Fig. 1). They obtained the dispersion relation

$$\epsilon \lambda f'(r) = \frac{\kappa^2 - (\omega - m\Omega)^2}{2\pi G n_0}, \tag{1}$$

where ϵ is the sign of $f'(r)$, $\Omega(r)$ is the angular frequency of the rotation of the disk, $n_0(r)$ is the equilibrium surface mass density, G is the gravitational constant, and κ is the epicyclic frequency defined by

$$\kappa^2(r) = 4\Omega^2\left(1 + \frac{r}{2\Omega}\frac{d\Omega}{dr}\right). \tag{2}$$

The physical meaning of the epicyclic frequency is explained as follows: Orbits of stars in a galaxy are usually not exact circles and always have a certain amount of randomness. If one of the perturbed orbits is observed on a framework rotating with the mean angular velocity around the center of the galaxy, it is known to be a small circle [1]. The frequency of such small cyclic motions that originate from randomness is called the epicyclic frequency.

In this paper, we extend Lin and Shu's analysis to the nonlinear regime and characterize the spiral structure as an asymptotic solitonlike nonlinear wave. Using the reductive perturbation theory [5], we derive a nonlinear equation for the envelope wave, which resembles a Schrödinger equation.

Within the framework of linear approximation, the pattern of galaxies is still undetermined, because any linear combination of spiral waves solves the linear equations. A specific structure stems from the nonlinear effect that yields coupling among linear modes and selects a special pattern which can sustain for a long term.

II. NONLINEAR DENSITY WAVE THEORY AND SOLITON STRUCTURE

We consider a galaxy whose mass density is concentrated on an infinitesimally thin disk. The set of the fluid equations (the mass conservation, the momentum equations, and Poisson's equation) reads, in the cylindrical polar coordinates (r,θ,z)

$$\frac{\partial n}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rnu) + \frac{\partial}{\partial \theta}(nv)\right] = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = \frac{\partial \phi}{\partial r}, \tag{4}$$

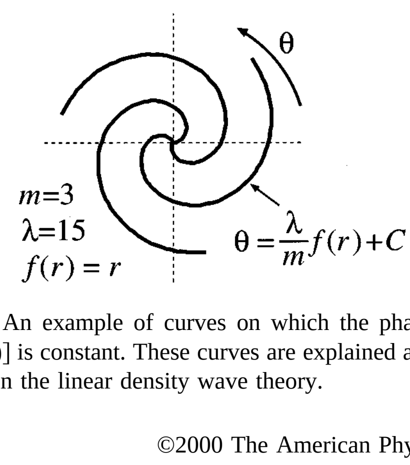

FIG. 1. An example of curves on which the phase of $\exp[\omega t - m\theta + \lambda f(r)]$ is constant. These curves are explained as the arms of the galaxy in the linear density wave theory.

1063-651X/2000/61(5)/5710(7)/$15.00

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}=\frac{1}{r}\frac{\partial \phi}{\partial \theta}, \tag{5}$$

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}=-4\pi G n(r,\theta)\delta(z), \tag{6}$$

where n is the surface mass density, u and v are the r component and the θ component of the fluid velocity, respectively, and ϕ is the negative of the gravitational potential. Mean fields are given by $n=n_0(r)$, $u=0$, $v=r\Omega(r)>0$. Here we normalize r and z by the mean wave length of the carrier wave in the radial direction $2\pi R/\lambda$, where R is the radial size of the galaxy and $\lambda \gg 1$ is a dimensionless constant, t by the period of the carrier wave $2\pi/\omega$, n by $n_0(0)$, u and v by the phase velocity $\omega R/\lambda$, ϕ by $\omega^2 R^2/\lambda^2$, and G by $\omega^2 R/2n_0(0)\lambda$.

Using a small parameter ε, we transform the independent variables (r,θ,z) into stretched variables (ξ,η,τ) as

$$\begin{aligned} \xi&=\varepsilon(r-Vt),\\ \eta&=\varepsilon^2\theta,\\ \tau&=\varepsilon^2 t, \end{aligned} \tag{7}$$

where V is a constant. The partial derivatives translates as

$$\begin{aligned} \frac{\partial}{\partial r}&=\varepsilon\frac{\partial}{\partial \xi},\\ \frac{\partial}{\partial \theta}&=\varepsilon^2\frac{\partial}{\partial \eta},\\ \frac{\partial}{\partial t}&=-\varepsilon V\frac{\partial}{\partial \xi}+\varepsilon^2\frac{\partial}{\partial \tau}. \end{aligned} \tag{8}$$

We expand the dependent variables around their mean values as

$$n=n_0+\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}n_l^{(n)}(\xi,\eta,\tau)\exp\{il[\omega t-m\theta+\lambda f(r)]\}, \tag{9}$$

$$u=\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}u_l^{(n)}(\xi,\eta,\tau)\exp\{il[\omega t-m\theta+\lambda f(r)]\}, \tag{10}$$

$$v=r\Omega+\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=-\infty}^{\infty}v_l^{(n)}(\xi,\eta,\tau)\exp\{il[\omega t-m\theta+\lambda f(r)]\}, \tag{11}$$

where, we assume $n_l^{(n)}$, $u_l^{(n)}$, and $v_l^{(n)}$ vary much slower than $\exp\{il[\omega t-m\theta+\lambda f(r)]\}$. The reality conditions on physical quantities demand

$$n_l^{(n)*}=n_{-l}^{(n)}, \tag{12}$$

$$u_l^{(n)*}=u_{-l}^{(n)}, \tag{13}$$

$$v_l^{(n)*}=v_{-l}^{(n)}, \tag{14}$$

where asterisks denote the complex conjugate.

We approximate the potential gradient as

$$\frac{\partial \phi}{\partial r}\simeq -r\Omega^2+2\pi G\epsilon\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=1}^{\infty}\mathrm{Re}(2in_l^{(n)}\exp\{il[\omega t-m\theta+\lambda f(r)]\}), \tag{15}$$

$$\frac{\partial \phi}{\partial \theta}\simeq 0, \tag{16}$$

which means that the phases of $\partial\phi/\partial r$ and $n^{(n)}$ are shifted by $\pi/2$. Equations (15) and (16) are in principle the same as Eq. (11) of Ref. [3], where the phase shift is given by the complex expression $\partial\phi/\partial r \propto in^{(n)}$. Note that the first term on the right-hand side of Eq. (15) is equivalent to the zeroth-order gravitational field.

Substituting Eqs. (7)–(16) into the fluid equations (3)–(5) gives

$$\begin{aligned} &\sum_{l=-\infty}^{\infty}\exp\{il[\omega t-m\theta+\lambda f(r)]\}\Bigg[\Bigg[il\omega-\varepsilon V\frac{\partial}{\partial \xi}+\varepsilon^2\frac{\partial}{\partial \tau}\Bigg]\sum_{n=1}^{\infty}\varepsilon^n n_l^{(n)}\\ &+\frac{\varepsilon^2}{V\tau}\left(1-\frac{\xi}{V\tau}\varepsilon\right)\left(n_0\sum_{n=1}^{\infty}\varepsilon^n u_l^{(n)}+\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}n_{l-l'}^{(n)}u_{l'}^{(n')}\right)+\left(il\lambda f'(r)+\varepsilon\frac{\partial}{\partial \xi}\right)\left(n_0\sum_{n=1}^{\infty}\varepsilon^n u_l^{(n)}\right)\\ &+\sum_{l'=-\infty}^{\infty}\left(il\lambda f'(r)+\varepsilon\frac{\partial}{\partial \xi}\right)\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}n_{l-l'}^{(n)}u_{l'}^{(n')}+\frac{\varepsilon^2}{V\tau}\left(1-\frac{\xi}{V\tau}\varepsilon\right)n_0\left(-ilm+\varepsilon^2\frac{\partial}{\partial \eta}\right)\sum_{n=1}^{\infty}\varepsilon^n v_l^{(n)}\\ &+\Omega\left(-ilm+\varepsilon^2\frac{\partial}{\partial \eta}\right)\sum_{n=1}^{\infty}\varepsilon^n n_l^{(n)}+\frac{\varepsilon^2}{V\tau}\left(1-\frac{\xi}{V\tau}\varepsilon\right)\sum_{l'=-\infty}^{\infty}\left\{\left(-ilm+\varepsilon^2\frac{\partial}{\partial \eta}\right)\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}n_{l-l'}^{(n)}v_{l'}^{(n')}\right\}\Bigg]=0, \end{aligned} \tag{17}$$

$$\sum_{l=-\infty}^{\infty} \exp\{il[\omega t-m\theta+\lambda f(r)]\}\Bigg[\Bigg[il\omega-\varepsilon V\frac{\partial}{\partial\xi}+\varepsilon^2\frac{\partial}{\partial\tau}\Bigg]\sum_{n=1}^{\infty}\varepsilon^n u_l^{(n)}+\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}u_{l-l'}^{(n)}\Bigg(il'\lambda f'(r)+\varepsilon\frac{\partial}{\partial\xi}\Bigg)u_{l'}^{(n')}$$

$$+\Omega\Bigg(-ilm+\varepsilon^2\frac{\partial}{\partial\eta}\Bigg)\sum_{n=1}^{\infty}\varepsilon^n u_l^{(n)}+\frac{\varepsilon^2}{V\tau}\Bigg(1-\frac{\xi}{V\tau}\varepsilon\Bigg)\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}v_{l-l'}^{(n)}\Bigg(-il'm+\varepsilon^2\frac{\partial}{\partial\eta}\Bigg)u_{l'}^{(n')}-2\Omega\sum_{n=1}^{\infty}\varepsilon^n v_l^{(n)}$$

$$-\frac{\varepsilon^2}{V\tau}\Bigg(1-\frac{\xi}{V\tau}\varepsilon\Bigg)\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}v_{l-l'}^{(n)}v_{l'}^{(n')}\Bigg]=2\pi G\epsilon\sum_{n=1}^{\infty}\varepsilon^n\sum_{l=1}^{\infty}\mathrm{Re}(2in_l^{(n)}\exp\{il[\omega t-m\theta+\lambda f(r)]\}), \tag{18}$$

$$\sum_{l=-\infty}^{\infty} \exp\{il[\omega t-m\theta+\lambda f(r)]\}\Bigg[\Bigg[il\omega-\varepsilon V\frac{\partial}{\partial\xi}+\varepsilon^2\frac{\partial}{\partial\tau}\Bigg]\sum_{n=1}^{\infty}\varepsilon^n v_l^{(n)}+\frac{\kappa^2}{2\Omega}\sum_{n=1}^{\infty}\varepsilon^n u_l^{(n)}$$

$$+\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}u_{l-l'}^{(n)}\Bigg(il'\lambda f'(r)+\varepsilon\frac{\partial}{\partial\xi}\Bigg)v_{l'}^{(n')}+\Omega\Bigg(-ilm+\varepsilon^2\frac{\partial}{\partial\eta}\Bigg)\sum_{n=1}^{\infty}\varepsilon^n v_l^{(n)}+\frac{\varepsilon^2}{V\tau}\Bigg(1-\frac{\xi}{V\tau}\varepsilon\Bigg)$$

$$\times\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}v_{l-l'}^{(n)}\Bigg(-il'm+\varepsilon^2\frac{\partial}{\partial\eta}\Bigg)v_{l'}^{(n')}+\frac{\varepsilon^2}{V\tau}\Bigg(1-\frac{\xi}{V\tau}\varepsilon\Bigg)\sum_{l'=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\varepsilon^{n+n'}u_{l-l'}^{(n)}v_{l'}^{(n')}\Bigg]=0. \tag{19}$$

Here we used a relation

$$\frac{1}{r}=\frac{1}{\varepsilon^{-1}\xi+\varepsilon^{-2}V\tau}=\frac{\varepsilon^2}{V\tau}\frac{1}{1+\varepsilon\xi/(V\tau)}\simeq\frac{\varepsilon^2}{V\tau}\Bigg(1-\varepsilon\frac{\xi}{V\tau}\Bigg). \tag{20}$$

Because the coefficients in Eqs. (17)–(19) are slow functions of t, r, and θ in comparison with the ''carrier'' $\exp\{il[\omega t-m\theta+\lambda f(r)]\}$, we can assume that each Fourier component is linearly independent, and hence, Eqs. (17)–(19) demand vanishing of every Fourier coefficients separately.

Now we separate terms in Eqs. (17)–(19) into each order of ε. The coefficients of the order of ε^1 read (see the Appendix)

$$il\omega n_l^{(1)}+il\lambda f'(r)n_0u_l^{(1)}-ilm\Omega n_l^{(1)}=0, \tag{21}$$

$$il\omega u_l^{(1)}-ilm\Omega u_l^{(1)}-2\Omega v_l^{(1)}=2\pi iG\epsilon n_l^{(1)}\mathrm{sgn}\, l, \tag{22}$$

$$il\omega v_l^{(1)}+\frac{\kappa^2}{2\Omega}u_l^{(1)}-ilm\Omega v_l^{(1)}=0 \tag{23}$$

for each Fourier component l. Equations (21)–(23) immediately yield the linear dispersion relation Eq. (1) and

$$u_{\pm1}^{(1)}=-pn_{\pm1}^{(1)}, \tag{24}$$

$$v_{\pm1}^{(1)}=\mp iqn_{\pm1}^{(1)}, \tag{25}$$

where

$$p=\frac{\omega-m\Omega}{\lambda f'(r)n_0}, \tag{26}$$

$$q=\frac{\kappa^2}{2\Omega\lambda f'(r)n_0}. \tag{27}$$

Note that we have assumed

$$u_l^{(1)}=v_l^{(1)}=n_l^{(1)}=0 \tag{28}$$

for $l\neq\pm1$, which means that the ''carrier wave'' is sinusoidal.

The coefficients of ε^2 in Eqs. (17)–(19) yield relations

$$il(\omega-m\Omega)n_l^{(2)}-V\frac{\partial n_l^{(1)}}{\partial\xi}+il\lambda f'(r)n_0u_l^{(2)}+\frac{\partial}{\partial\xi}(n_0u_l^{(1)})$$

$$+il\lambda f'(r)\sum_{l'=-\infty}^{\infty}n_{l'}^{(1)}u_{l-l'}^{(1)}=0, \tag{29}$$

$$il(\omega-m\Omega)u_l^{(2)}-V\frac{\partial u_l^{(1)}}{\partial\xi}-2\Omega v_l^{(2)}$$

$$+\sum_{l'=-\infty}^{\infty}il'\lambda f'(r)u_{l-l'}^{(1)}u_{l'}^{(1)}=2\pi iG\epsilon n_l^{(2)}\mathrm{sgn}\, l, \tag{30}$$

$$il(\omega-m\Omega)v_l^{(2)}-V\frac{\partial v_l^{(1)}}{\partial\xi}+\frac{\kappa^2}{2\Omega}u_l^{(2)}$$

$$+\sum_{l'=-\infty}^{\infty}il'\lambda f'(r)u_{l-l'}^{(1)}v_{l'}^{(1)}=0, \tag{31}$$

for each l. Substituting Eqs. (24)–(28) and $l=2$ into Eqs. (29)–(31) gives

$$u_2^{(2)}=b_1n_1^{(1)2}, \tag{32}$$

$$v_2^{(2)}=ib_2n_1^{(1)2}, \tag{33}$$

$$n_2^{(2)}=b_3n_1^{(1)2}, \tag{34}$$

where

$$b_1=\frac{(\omega-m\Omega)(-\kappa^2+4\pi G\epsilon\lambda f'(r)n_0)}{\lambda f'(r)n_0^2[4(\omega-m\Omega)^2-\kappa^2+4\pi G\epsilon\lambda f'(r)n_0]}, \tag{35}$$

$$b_2=-\frac{\kappa^2(\omega-m\Omega)^2}{\lambda f'(r)\Omega n_0^2[4(\omega-m\Omega)^2-\kappa^2+4\pi G\epsilon\lambda f'(r)n_0]}, \tag{36}$$

$$b_3=\frac{4(\omega-m\Omega)^2}{n_0[4(\omega-m\Omega)^2-\kappa^2+4\pi G\epsilon\lambda f'(r)n_0]}. \tag{37}$$

Similarly substituting Eqs. (24)–(28) and $l=1$ into Eqs. (29) and (31) yield

$$\begin{bmatrix} i(\omega-m\Omega) & i\lambda f'(r)n_0 & 0 \\ -2\pi iG\epsilon & i(\omega-m\Omega) & -2\Omega \\ 0 & \dfrac{\kappa^2}{2\Omega} & i(\omega-m\Omega)\end{bmatrix}\begin{bmatrix} n_1^{(2)} \\ u_1^{(2)} \\ v_1^{(2)}\end{bmatrix} =\begin{bmatrix} (V+n_0p)\dfrac{\partial n_1^{(1)}}{\partial\xi} \\ -pV\dfrac{\partial n_1^{(1)}}{\partial\xi} \\ -iqV\dfrac{\partial n_1^{(1)}}{\partial\xi}\end{bmatrix}. \tag{38}$$

Since the determinant of the matrix in Eq. (38) is zero due to Eqs. (21)–(23), there are no solutions to Eq. (38) unless

$$V=\frac{\pi G n_0\epsilon}{\omega-m\Omega}. \tag{39}$$

Equation (39) determines the group velocity of the nonlinear wave. It might seem a contradiction that the right-hand side of Eq. (39) is a function of r, while V is a constant. However, Eq. (39) is locally justified by the fact that the mean fields are slow functions of r.

Under the condition (39), solutions to Eq. (38) cannot be determined uniformly, because the determinant of the matrix in the left-hand side is zero. Hence we assume

$$n_1^{(2)}=0. \tag{40}$$

Equations (38) and (40) yield

$$u_1^{(2)}=\frac{1}{i\lambda f'(r)n_0}\frac{\partial}{\partial\xi}[(V+n_0p)n_1^{(1)}]-\frac{\omega-m\Omega}{\lambda f'(r)n_0}n_1^{(2)}, \tag{41}$$

$$v_1^{(2)}=\frac{1}{\omega-m\Omega}\left[\left(-V\frac{\partial q}{\partial\xi}+q\frac{\partial}{\partial\xi}(n_0p)\right)n_1^{(1)}+n_0pq\frac{\partial n_1^{(1)}}{\partial\xi}\right] -iqn_1^{(2)}. \tag{42}$$

Furthermore, we substitute $l=0$ into Eqs. (30) and (31) and use Eqs.(12) and (24)-(28). Then the following equations are obtained:

$$u_0^{(2)}=\frac{2p}{n_0}|n_1^{(1)}|^2, \tag{43}$$

$$v_0^{(2)}=0. \tag{44}$$

The coefficients of ε^3 in Eqs. (17)–(19) read

$$il(\omega-m\Omega)n_l^{(3)}-V\frac{\partial n_l^{(2)}}{\partial\xi}+\frac{\partial n_l^{(1)}}{\partial\tau}+\frac{1}{V\tau}n_0u_l^{(1)}+il\lambda f'(r)n_0u_l^{(3)}+\frac{\partial}{\partial\xi}(n_0u_l^{(2)}) +\sum_{l'=-\infty}^{\infty}\left\{il\lambda f'(r)(n_{l'}^{(1)}u_{l-l'}^{(2)}+n_{l'}^{(2)}u_{l-l'}^{(1)})+\frac{\partial}{\partial\xi}(n_{l'}^{(1)}u_{l-l'}^{(1)})\right\}-\frac{ilmn_0}{V\tau}v_l^{(1)}+\Omega\frac{\partial n_l^{(1)}}{\partial\eta}=0, \tag{45}$$

$$il(\omega-m\Omega)u_l^{(3)}-V\frac{\partial u_l^{(2)}}{\partial\xi}+\frac{\partial u_l^{(1)}}{\partial\tau}+\sum_{l'=-\infty}^{\infty}\left\{il'\lambda f'(r)(u_{l-l'}^{(1)}u_{l'}^{(2)}+u_{l-l'}^{(2)}u_{l'}^{(1)})+u_{l-l'}^{(1)}\frac{\partial u_{l'}^{(1)}}{\partial\xi}\right\}+\Omega\frac{\partial u_l^{(1)}}{\partial\eta}-2\Omega v_l^{(3)}=2\pi iG\epsilon n_l^{(3)}, \tag{46}$$

$$il(\omega-m\Omega)v_l^{(3)}-V\frac{\partial v_l^{(2)}}{\partial\xi}+\frac{\partial v_l^{(1)}}{\partial\tau}+\frac{\kappa^2}{2\Omega}u_l^{(3)}+\sum_{l'=-\infty}^{\infty}\left\{il'\lambda f'(r)(u_{l-l'}^{(1)}v_{l'}^{(2)}+u_{l-l'}^{(2)}v_{l'}^{(1)})+u_{l-l'}^{(1)}\frac{\partial v_{l'}^{(1)}}{\partial\xi}\right\}+\Omega\frac{\partial v_l^{(1)}}{\partial\eta}=0, \tag{47}$$

for each l. Substituting $l=0$ into Eq. (45) and using Eqs. (12) and (24) give

$$\frac{\partial}{\partial\xi}(-Vn_0^{(2)}+n_0u_0^{(2)}-2p|n_1^{(1)}|^2)=0. \tag{48}$$

Substituting Eq. (43) into Eq. (48) gives

$$\frac{\partial n_0^{(2)}}{\partial \xi}=0, \tag{49}$$

which can be immediately integrated as

$$n_0^{(2)}=\psi(\eta,\tau), \tag{50}$$

where $\psi(\eta,\tau)$ is an arbitrary function of η and τ. Here we assume $\psi\equiv 0$ to obtain

$$n_0^{(2)}=0. \tag{51}$$

For $l=1$, Eqs. (45)–(47) yield

$$i(\omega-m\Omega)n_1^{(3)}+i\lambda f'(r)n_0u_1^{(3)}+g_1=0, \tag{52}$$

$$i(\omega-m\Omega)u_1^{(3)}-2\Omega v_1^{(3)}-2\pi iG\epsilon n_1^{(3)}+g_2=0, \tag{53}$$

$$i(\omega-m\Omega)v_1^{(3)}+\frac{\kappa^2}{2\Omega}u_1^{(3)}+g_3=0, \tag{54}$$

where

$$\begin{aligned}g_1=&-V\frac{\partial n_1^{(2)}}{\partial \xi}+\frac{\partial n_1^{(1)}}{\partial \tau}+\frac{n_0}{V\tau}u_1^{(1)}+\frac{\partial}{\partial \xi}(n_0u_1^{(2)})-\frac{in_0m}{V\tau}v_1^{(1)}\\&+\Omega\frac{\partial n_1^{(1)}}{\partial \eta}+i\lambda f'(r)(n_1^{(1)}u_0^{(2)}+n_0^{(2)}u_1^{(1)}+n_{-1}^{(1)}u_2^{(2)}\\&+n_2^{(2)}u_{-1}^{(1)}),\end{aligned} \tag{55}$$

$$\begin{aligned}g_2=&-V\frac{\partial u_1^{(2)}}{\partial \xi}+\frac{\partial u_1^{(1)}}{\partial \tau}+\Omega\frac{\partial u_1^{(1)}}{\partial \eta}+i\lambda f'(r)(u_{-1}^{(1)}u_2^{(2)}\\&+u_0^{(2)}u_1^{(1)}),\end{aligned} \tag{56}$$

$$\begin{aligned}g_3=&-V\frac{\partial v_1^{(2)}}{\partial \xi}+\frac{\partial v_1^{(1)}}{\partial \tau}+\Omega\frac{\partial v_1^{(1)}}{\partial \eta}+i\lambda f'(r)(2u_{-1}^{(1)}v_2^{(2)}\\&+u_0^{(2)}v_1^{(1)}-u_2^{(2)}v_{-1}^{(1)}).\end{aligned} \tag{57}$$

Under the condition given by Eq. (1), Eqs. (52)–(54) are not independent with respect to $n_1^{(3)}$, $u_1^{(3)}$, and $v_1^{(3)}$. Equation (54) times $2i\Omega/(\omega-m\Omega)$ minus Eq. (52) times $2\pi G\epsilon/(\omega-m\Omega)$ becomes

$$\begin{aligned}&i(\omega-m\Omega)u_1^{(3)}-2\Omega v_1^{(3)}-2\pi iG\epsilon n_1^{(3)}+\frac{2i\Omega g_3-2\pi G\epsilon g_1}{\omega-m\Omega}\\&=0,\end{aligned} \tag{58}$$

where the dispersion relation Eq. (1) has been used. Comparing Eqs. (53) and (58), we obtain

$$(\omega-m\Omega)g_2=-2\pi G\epsilon g_1+2i\Omega g_3. \tag{59}$$

Finally, we substitute Eqs. (24)–(28), (32)–(34), (40)–(44), (51), and (55)–(57) into Eq. (59) and assume that p, q, $f'(r)$, Ω, and n_0 vary much slower than $n_1^{(1)}$. Then, we obtain

$$i\left(\frac{\partial}{\partial \tau}+\Omega\frac{\partial}{\partial \eta}\right)n_1^{(1)}+d_1\frac{\partial^2 n_1^{(1)}}{\partial \xi^2}+\frac{id_2}{\tau}n_1^{(1)}+d_3|n_1^{(1)}|^2n_1^{(1)}=0, \tag{60}$$

where

$$d_1=\frac{2V(\omega-m\Omega)-2\pi G\epsilon n_0+V^2\lambda f'(r)}{2(\omega-m\Omega)\lambda f'(r)}, \tag{61}$$

$$d_2=\frac{\pi G\epsilon n_0^2\lambda f'(r)(mq+p)}{(\omega-m\Omega)^2V}, \tag{62}$$

$$d_3=-\frac{3(\omega-m\Omega)\kappa^2}{[\kappa^2+2(\omega-m\Omega)^2]n_0^2}. \tag{63}$$

Taking the limit of $\tau\to\infty$, Eq. (60) reduces into the nonlinear Schrödinger equation, which describes envelope solitons [6]. We thus see that the spiral structure of galaxies approaches to a solitonlike state.

Note that $\exp\{il[\omega t-m\theta+\lambda f(r)]\}$ does not express an observable wave pattern any longer. In other words, the variable m is not a number of arms of a galaxy any longer (which used to be in the linear theory by Lin and Shu [3]). Instead, $\exp\{il[\omega t-m\theta+\lambda f(r)]\}$ expresses the carrier of the nonlinear wave here.

III. NUMERICAL SOLUTIONS

Although Eq. (60) reduces into the nonlinear Schrödinger equation in the limit of $\tau\to\infty$, no exact solutions to Eq. (60) have been found for $d_2\neq 0$. In this section, we study the solutions for finite τ numerically.

If $d_2=0$, the analytic solution to Eq. (60) is well known to be

$$\begin{aligned}n_1^{(1)}=&A\,\mathrm{sech}\left[A\sqrt{\frac{d_3}{2d_1}}(\xi+V_1\tau+\xi_0)\right]\\&\times\exp\left[i\left(\frac{d_3A^2}{2}-\frac{V_1^2}{4d_1}\right)\tau-i\frac{V_1}{2d_1}(\xi+\xi_1)\right]\end{aligned} \tag{64}$$

for $d_1d_3>0$ or

$$\begin{aligned}n_1^{(1)}=&A\tanh\left[A\sqrt{-\frac{d_3}{2d_1}}(\xi+V_1\tau+\xi_0)\right]\\&\times\exp\left[i\left(d_3A^2-\frac{V_1^2}{4d_1}\right)\tau-i\frac{V_1}{2d_1}(\xi+\xi_1)\right]\end{aligned} \tag{65}$$

for $d_1d_3<0$, where A, V_1, ξ_0, and ξ_1 are arbitrary constants. The constant A determines the amplitude and the width of the soliton, V_1 (and V) determine(s) the group velocity, and ξ_0 and ξ_1 determine the initial positions of the soliton and the carrier wave, respectively. Equations (64) and (65) correspond to the bright and dark solitons, respectively.

Figure 2 shows an analytic solution for $d_2=0$ given by Eq. (64). Here $d_1=d_3=1$, $A=2$, $V_1=-10$, $\xi_0=6$, $\xi_1=2.76$, and the initial time $\tau_0=1$. We can see that an envelope soliton propagates in the ξ direction without change of

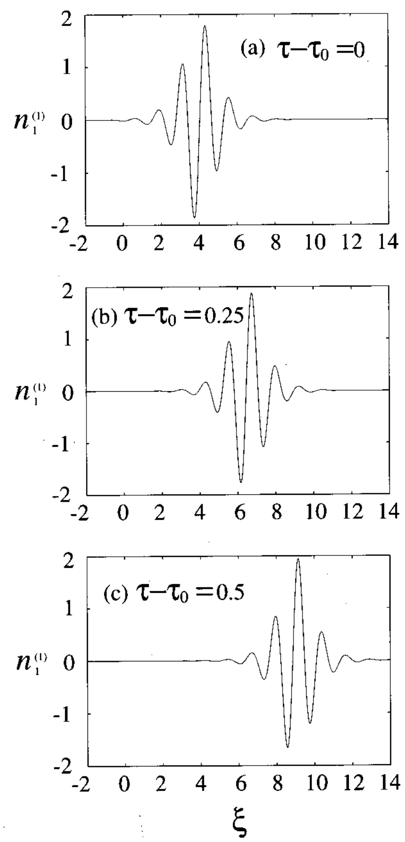

FIG. 2. An analytic solution for $d_2=0$ given by Eq. (64). Here $d_1=d_3=1$, $A=2$, $V_1=-10$, $\xi_0=6$, $\xi_1=2.76$, and $\tau_0=1$.

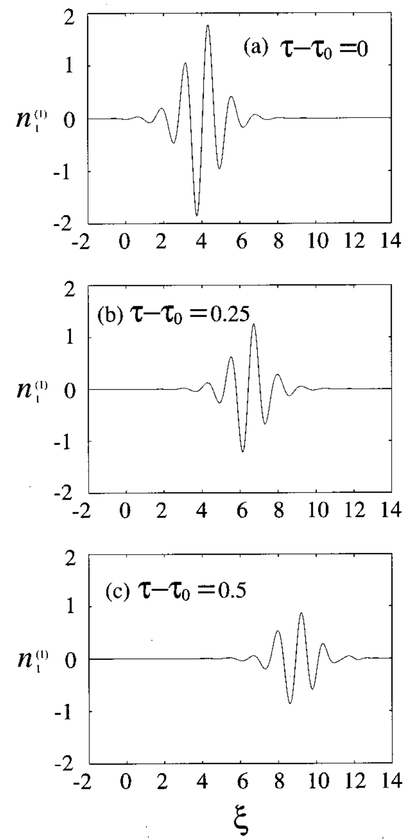

FIG. 3. A numerical solution for $d_2=1$. Here $\tau_0=0.25$. The initial position of the wave is the same as Fig. 2. We can see that the amplitude decreases as the solitary wave propagates in the ξ direction.

its shape. On the other hand, Figs. 3 and 4 show numerical solutions to Eq. (60) for $d_2=1$. The initial time $\tau_0=0.5$ in Fig. 3 and $\tau_0=0.1$ in Fig. 4. The initial positions of the soliton and the carrier wave in Figs. 3 and 4 are the same as Fig. 2. We can see from these figures that the amplitude of the solitary waves decreases as it propagates in the ξ direction. This result is quite natural, because the wave propagating in the ξ direction, which corresponds to the radial direction, decreases its energy density [the $1/\tau$ term in Eq. (60) comes from the $1/r$ terms in Eqs. (3)–(5)].

IV. CONCLUDING REMARKS

We have derived a nonlinear Schrödinger-type equation that describes a spiral structure of a galaxy. As is well known to plasma physics, the Langmuir wave, which is a density wave propagating on electrons, can generate an envelope soliton that is described by a nonlinear Schrödinger equation. A galaxy is a gravitational many-body system, while a plasma is an electromagnetic many-body system, which, in the electrostatic limit, obeys the same set of equations. Hence, the fact that the galactic density wave has a soliton-like structure can be naturally deduced. A unique feature of the present formalism is that a thin-disk geometry is considered and the soliton-like structure of a gravitational medium is obtained.

Galaxies resemble plasmas in that their components interact with each other through a long range force produced by the particles themselves. In particular, single-species non-neutral plasmas such as pure electron plasmas are quite similar to galaxies except that electrons are repulsive while stars are attractive. Because of this nature of galaxies and non-neutral plasmas, they exhibit similar collective phenomena such as vortices [7].

The behavior of single-species non-neutral plasmas is often analyzed under the assumption that they have an infinite length in the axial direction, while galaxies are often assumed to have a zero thickness. The advantage of the infinite-length analysis is that the Poisson's equation becomes two dimensional, which makes our problem completely two dimensional. Then the analysis is greatly simplified. It is meaningful to analyze the behavior of galaxies in such a way because of its simplicity. Such an analysis is to be published elsewhere.

ACKNOWLEDGMENT

The authors are grateful to Dr. Tomoya Tatsuno for his useful discussions.

APPENDIX

We show that the following two equations are equivalent:

$$\sum_{l=-\infty}^{\infty} \exp(il\varphi)(X_l^{(n)}+ilY_l^{(n)})+\sum_{l=1}^{\infty} \mathrm{Re}[2iZ_l^{(n)}\exp(il\varphi)] = 0 \quad (\forall\varphi), \tag{A1}$$

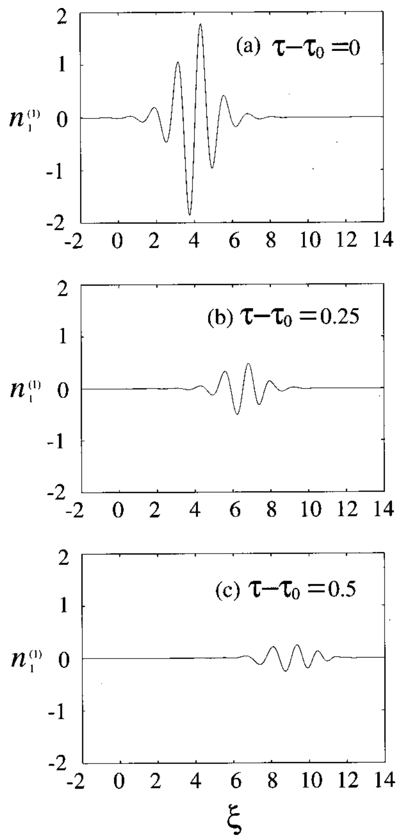

FIG. 4. A numerical solution for $d_2=1$. Here $\tau_0=0.1$.

$$X_l^{(n)}+ilY_l^{(n)}+iZ_l^{(n)}\mathrm{sgn}\, l=0 \quad (\forall l), \tag{A2}$$

where $X_l^{(n)}$, $Y_l^{(n)}$, and $Z_l^{(n)}$ are complex constants or slowly varying functions of φ that satisfy the reality condition

$$X_l^{(n)*}=X_{-l}^{(n)}, \tag{A3}$$

$$Y_l^{(n)*}=Y_{-l}^{(n)}, \tag{A4}$$

$$Z_l^{(n)*}=Z_{-l}^{(n)}. \tag{A5}$$

Equation (A1) can be written as

$$\sum_{l=-\infty}^{\infty}[(X_{l\mathrm{r}}^{(n)}-lY_{l\mathrm{i}}^{(n)})\cos(l\varphi)-(X_{l\mathrm{i}}^{(n)}+lY_{l\mathrm{r}}^{(n)})\sin(l\varphi)]$$
$$+i\sum_{l=-\infty}^{\infty}[(X_{l\mathrm{i}}^{(n)}+lY_{l\mathrm{r}}^{(n)})\cos(l\varphi)$$
$$+(X_{l\mathrm{r}}^{(n)}-lY_{l\mathrm{i}}^{(n)})\sin(l\varphi)]-\sum_{l=1}^{\infty}[2Z_{l\mathrm{i}}^{(n)}\cos(l\varphi)$$
$$+2Z_{l\mathrm{r}}^{(n)}\sin(l\varphi)]=0, \tag{A6}$$

where subscripts r and i denote the real part and imaginary part, respectively. The second term of the left-hand side of Eq. (A6) vanishes because of the reality condition Eqs. (A3)–(A5). Then Eq. (A6) becomes

$$\sum_{l=0}^{\infty}[2(X_{l\mathrm{r}}^{(n)}-lY_{l\mathrm{i}}^{(n)}-Z_{l\mathrm{i}}^{(n)})\cos(l\varphi)$$
$$-2(X_{l\mathrm{i}}^{(n)}+lY_{l\mathrm{r}}^{(n)}+Z_{l\mathrm{r}}^{(n)})\sin(l\varphi)]=0. \tag{A7}$$

Thus,

$$X_{l\mathrm{r}}^{(n)}-lY_{l\mathrm{i}}^{(n)}-Z_{l\mathrm{i}}^{(n)}=0, \tag{A8}$$

$$X_{l\mathrm{i}}^{(n)}+lY_{l\mathrm{r}}^{(n)}+Z_{l\mathrm{r}}^{(n)}=0, \tag{A9}$$

for each $l>0$. Equations (A8) and (A9) are identical to

$$X_l^{(n)}+ilY_l^{(n)}+iZ_l^{(n)}=0 \quad (l>0). \tag{A10}$$

Taking complex conjugate on Eq. (A10) gives

$$X_{-l}^{(n)}-ilY_{-l}^{(n)}-iZ_{-l}^{(n)}=0 \quad (l>0)$$
$$\Leftrightarrow\ X_l^{(n)}+ilY_l^{(n)}-iZ_l^{(n)}=0 \quad (l<0). \tag{A11}$$

Equations (A10) and (A11) are written as

$$X_l^{(n)}+ilY_l^{(n)}+iZ_l^{(n)}\mathrm{sgn}\, l=0 \quad (\forall l). \tag{A12}$$

[1] G. Bertin and C.C. Lin, *Spiral Structure in Galaxies: A Density Wave Theory* (The MIT Press, Massachusetts, 1996).
[2] S. Chandrasekhar, *Principles of Stellar Dynamics* (Dover, New York, 1960), p. 184
[3] C.C. Lin and F.H. Shu, Astrophys. J. **140**, 646 (1964).
[4] C.C. Lin and F.H. Shu, Proc. Natl. Acad. Sci. USA **55**, 229 (1966).
[5] A. Jeffrey and T. Kawahara, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982), p. 70.
[6] A. Jeffrey and T. Taniuti, *Non-linear Wave Propagation: with Applications to Physics and Magnetohydrodynamics* (Academic Press, New York, 1964).
[7] C.F. Driscoll and K.S. Fine, Phys. Fluids B **2**, 1359 (1990).